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Recovery Simulator and Analysis Formulation

Mathematical Framework for Enhanced Resilience and Resource Allocation

October 2025

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Executive Summary

This report introduces recovery simulator and analysis (RSA), a framework aimed at enhancing the resilience of electrical grids post-disruption. The RSA model leverages an optimization problem formulation that focuses on maximizing the load served (or optionally customers served) through a coordinated and cooptimized recovery of non-black start generation, transmission lines, feeders and substations subject to labor budget constraints. By integrating advanced linear programming techniques, the simulator selects efficient reocovery pathways, optimizing both short-term and long-term grid recovery strategies.

The mathematical framework guides decision-making through a comprehensive evaluation of potential recovery actions, factoring in the trade-offs between labor constraints and load (or optionally customer) restoration efficacy. This enables grid operators to simulate diverse outage scenarios and delineate optimal recovery pathways, thereby prioritizing critical repair tasks and ensuring resource allocation is both economical and effective. The intended use case of RSA is to allow planners to explore many recovery scenarios quickly and determine assets most critical across a wide range of scenarios, and therefore strong candidates for hardening or additional investment. RSA might also be used in an operational setting, following a single event, for exploring efficient recovery pathways.

Contents

Executive Summary iv

1.0 Introduction 1

2.0 Indices 2

3.0 Sets 3

4.0 Parameters 4

5.0 Decision Variables 5
- 5.1 Continuous 5
- 5.2 Binary 5

6.0 Objective Function 6
- 6.1 Maximize Load Served 6
- 6.2 Alternative Objective for Client Recovery 6

7.0 Problem Formulation: Optimization Constraints 8
- 7.1 Power Balance 8
- 7.2 Generation Limits (Traditional) 8
- 7.3 Generation Limits (Variable) 8
- 7.4 Transmission and Subtransmission Model 8
  - 7.4.1 Non-Damaged Lines 8
  - 7.4.2 Damaged Lines - Recovery Candidates 9
- 7.5 Transmission and Subtransmission Recovery Budget Equations 9
- 7.6 Substation (Bus) Outage Model 10
- 7.7 Feeder Model 10
- 7.8 Thermal Generation Considering Black Start and Non-Black Start Units. 11

8.0 Appendix: Simplified Reformulation of a Max Function 13

1.0 Introduction

The RSA framework propses a framework for electrical grid resilience via an optimization problem formulation that seeks to maximize the load served (or optionally customers served) following an event of grid disruption while maintaining the operational and physical capacities of the assets. Specifically, this formulation accounts for coordinated and cooptimized recovery of non-black start generation, transmission lines, feeders and substations subject to labor budget constraints. By mathematically modeling the grid's operational state, RSA utilizes optimization techniques to map outage scenarios to feasible recovery pathways. The objective function is structured to maximize load served (or alternatively clients served) by cooptimizing asset recovey subject to labor and physics based constraints The overall goal being to identify and prioritize bringing high impact low repair effort assets online first. This systematic approach enables RSA to dynamically allocate repair resources by weighing the effectiveness of each recovery action against its associated costs, thereby optimizing the grid's reliability in an efficient manner.

To achieve this optimization, RSA defines a set of decision variables representing the state of generation units, transmission lines, feeders, substations. These variables are subject to constraints derived from the grid's topology and repair budget limitations. The constraints ensure that the repair actions align with the maximum deployable work crew days and resource availability, therby maintaining feasibility in real-world scenarios. RSA incorporates a set of mixed-integer linear equations to delineate the repair sequence and ensures that each decision enhances the load served (or alternatively clients served). Additionally, the RSA formulation objective includes penalty terms for any unserved loads, and rewards for critical loads encouraging configurations that minimize the impact on end-users and critical service infrastructure. When solving the RSA optimization model, varying levels of grid stress can be simulated and multiple recovery scenarios are evaluated which provides insights into effective restoration strategies.

2.0 Indices

t: recovery time period
b: system bus
k: generation technology
u: generator id, used to differentiate multiple units at the same bus
w: load id, used to differentiate multiple loads at the same bus
y: line id, used to differentiate multiple lines connected between the same two buses
z: feeder id, used to differentiate different feeders connected to the same bus
$g_{b,u,k}$: generator at bus b, with generator ID of u, and technology k. Nomenclature often simplified to g
$a_{b,b'}$: arc (line) between buses b and b'

3.0 Sets

B: set of all buses b
G: set of all generators g
G^R: set of all renewable generators ($G^R \subset G$)
G^T: set of all traditional generators ($G^T \subset G$)
T: set of all recovery time periods t
A: set of arcs (lines) between buses
A^D: set of arcs (lines) between buses that are initially damaged at the start of the simulation $A^D \subset A$
A^{ND}: set of arcs (lines) between buses that begin the simulation in service and have no damage $A^{ND} \subset A$
B^D: set of buses that are initially damaged and out of service at the start of the simulation $B^D \subset B$
B^{ND}: set of buses that begin the simulation in service and have no damage $B^{ND} \subset B$
K: set of all generation technologies k
U: set of all generator ids u
W: set of all load ids w
Y: set of all branch ids y
Z: set of all feeder ids z

4.0 Parameters

$D_{t,b,w}$: demand at time period t, bus b, id w. (MW)
$C_{t,g_{b,u,k}}$: normalized output of generator g, unit id u, technology k. (unitless)
$P^{min}_{g_{b,u,k}}$: min power output of generator g. (MW)
$P^{max}_{g_{b,u,k}}$: max power output of generator g. (MW)
$F^{min}_{a_{b,b'},y}$: capacity of line along arc a in period t from b' to b with branch id y. (MW)
$F^{max}_{a_{b,b'},y}$: capacity of line along arc a in period t from b to b' with branch id y. (MW)
$TRT^{wcd}_{a,y}$: Transmission (and subtransmission) recovery time of arc a, branch id y in work crew days. (work crew days)
$RB^{wcd}_{TS,t}$: Recovery time budget in work crew days (for transmission and sub-transmission) in recovery time period t. (work crew days)
$RB^{num}_{TS,t}$: Recovery time budget in number of transmission-subtransmission lines in recovery time period t. (number of branches)
$RB^{wcd}_{F,t}$: Recovery time budget in work crew days (for feeders) in recovery time period t. (work crew days)
$RB^{wcd}_{B,t}$: Recovery time budget in work crew days (for buses) in recovery time period t. (work crew days)
$X_{a_{b,b'},y}$: Reactance of line a. (Ohms)
M: A large number - must be large enough such that when $s_{t,a,y}$ = 0 in equation (14), equation (14) is non-binding.
M': Must be large enough such that when $k^0_{b,t}$ in (35) (or $k_{b,t,\tau}$ in (36)) is zero, (35) (or (36)) is nonbinding.
$FL_{t,b,z}$: Feeder load time period t, bus b, feeder z. (MW)
$FRT_{b,z}$: Feeder recovery time at bus b, feeder z. (work crew days)
$N_{b,z}$: Total clients at bus b, feeder z. (clients)
B: Reference bus where angle is set to zero. (unitless)
$C_{b,z}$: Feeder criticality scaling factor; non-critical $=$ 1, critical $>$ 1 (unitless)
LC_b: Lines connected to bus b. (unitless)
$TL_{t,bb,z}$: Aggregated load to transmission bus. (MW)

5.0 Decision Variables

5.1 Continuous

$ens_{t,b}$: Energy not served at at time period t, bus b. (MW)
$ll_{t,b}$: Load lift at time period t, bus b. Is necessary if generator exists at bus disconnected from system with $p_{t,g_{b,u,k}} > 0$ and there is no load to absorb the generation. (MW)
$f_{a_{b,b'},y,t}$: Power flow from b to b' over arc a on branch id y at time period t. Simplified to $f_{a,y,t}$ where possible (MW).
$p_{t,g_{b,u,k}}$: Real power output of generator g, at bus b, unit id u, technology k at time t. (MW)
$\rho_{b,t}$: Status of bus b at timepoint t. (unitless)
$\theta_{t,b}$: Angle at time t at bus b. (radians)
$frt_{t,b,z}$: Time consumed in time step t by work crews to recovery feeder z at bus b. (work crew days)
$trt^{wcd}_{t,a,y}$: Time consumed in time step t by work crews to recovery branch a, branch id y. (work crew days)
$n_{t,b,z}$: clients recovered in time step t at feeder z at bus b. (clients)
$a_{b,t}$: ratio of incoming flow at bus with non-black start unit to non-black start unit required cranking power at bus b in time period t. (unitless)
$max^{a}_{b,t}$: Maximum $a_{b,t}$ for $\tau \leq t$. (unitless)

5.2 Binary

$s_{t,a,y}$: Status of arc a, branch id y at timepoint t. (unitless)
$\eta_{t,g,u}$: Status of unit g, unit id u at timepoint t. (unitless)
$k^{0}_{b,t}$: Variable indicating if 0 is maximum value in max function at time t, bus b. (unitless)
$k_{b,t,\tau}$: Variable indicating if $a_{b,\tau}$ is maximum value in max function for $\tau \leq t$, at bus b. (unitless)

6.0 Objective Function

6.1 Maximize Load Served

The objective of the optimization formulation is to maximize load served across the system. The first summation contains the slack variables energy not served $ens_{t,b}$ and load lift $ll_{t,b}$ used in the power balance equation to ensure the problem remains feasible. The second summation contains load with a feeder representation; this term is used to represent damaged feeders with non-zero recovery time ($FRT_{b,z} > 0$). In the summation term, $frt_{t,b,z}$ represents the cumulative number of work crew days expended on recovering feeder z at bus b by time step t while $FRT_{b,z}$ represents the total recovery time of feeder z at bus b. Because $FL_{t,b,z}$ represents the total load at feeder z, bus b, timestep t, $\frac{frt_{t,b,z}}{FRT_{b,z}}$ can be interpreted as the fraction of load $FL_{t,b,z}$ that has been recovered at feeder z, bus b in time step t (feeders can factionally recovery across multiple time steps). Finally, undamaged feeders at bus b, feeder z ($FL_{t,b,z}$ with $FRT_{b,z} = 0$) as well as transmission load buses with no feeder mapping (due to datasets being used) ($D_{t,b,w}$ such that $\sum_{b,z} FL_{t,b,z} = 0$) are simply lumped into a transmission load term $TL_{t,b}$. Aggregated transmission load $TL_{t,b}$ at time period t, bus b, is defined in (2).

$$\begin{aligned} \text{max} \Bigg[&- \sum_{t,b} \Big(ens_{t,b} + ll_{t,b} \Big) \\ &+ \sum_{t,b,z:FRT_{b,z}>0} \Big(C_{b,z} \cdot \frac{frt_{t,b,z}}{FRT_{b,z}} \cdot FL_{t,b,z} \Big) + \sum_{t,b} TL_{t,b} \Bigg] \end{aligned} \tag{1}$$

Where,

$$TL_{t,b} = \sum_{z:FRT_{b,z}=0} FL_{t,b,z} + \sum_{w:\sum_z FL_{t,b,z}=0} D_{t,b,w} \tag{2}$$

6.2 Alternative Objective for Client Recovery

In order to recover clients, the relationship between clients and load is defined as

$$\frac{frt_{t,b,z}}{FRT_{b,z}} = \frac{n_{t,b,z}}{N_{b,z}} \quad \forall\, t \in T, \forall\, b \in B, \forall\, z \in Z \tag{3}$$

This ensures that all clients are recovered when all load is recover and that if no load is recovered, no clients are recovered. The number of clients recovered at a feeder $n_{t,b,z}$ is then bounded by the total number of clients at the feeder $N_{b,z}$.

$$0 \leq n_{t,b,z} \leq N_{b,z} \quad \forall\, t \in T, \forall\, b \in B, \forall\, z \in Z. \tag{4}$$

Finally, the objective is replace by

$$\max \Bigg[-\sum_{t,b} \Big(ens_{t,b} + ll_{t,b} \Big) \\ + \sum_{t,b,z:FRT_{b,z}>0} \epsilon \cdot \Big(C_{b,z} \cdot \frac{frt_{t,b,z}}{FRT_{b,z}} \cdot N_{b,z} \Big) + \sum_{t,b} TL_{t,b} \Bigg] \tag{5}$$

This is the exact same objective as that used to maximize load served (1) except the second summation term representing feeder load has been replaced by the number of clients, or $n_{b,z} = \frac{frt_{t,b,z}}{FRT_{b,z}} \cdot N_{b,z}$. Because the replacement of load with clients on feeders causes the optimization to be multi-objective (different terms have different units), we include a scaling factor ϵ to get the clients and loads on the system in the objective to approximately the same order of magnitude. Without this scaling factor, a small MIP gap could result could result in a relatively large amount of load being un-served since the number of clients is generally much larger than the quantity of load (in MW).

7.0 Problem Formulation: Optimization Constraints

7.1 Power Balance

Constraint (6) is the power balance equation for real power at each time step t and bus b. In the first row of the (6), the sum over $p_{t,g_{b,u}}$ represents all the power generated by generators at bus b, timestep t, $ens_{t,b}$ represents energy not served at bus b, timestep t, and $f_{a_{b,b'},y,t}$ is the power flow between buses b and b' over arc a, branch id y at time t. In row 2 of (6) the summation represents the amount of feeder load recovered for feeders that are damaged ($FRT_{b,z} > 0$) while the second term outside the summation, $TL_{t,b}$, denotes a transmission load comprised of load at undamaged feeders at bus b, timesstep t and transmission load buses with no feeder mapping. $TL_{t,b}$ is defined in (2).

$$\begin{aligned} &\sum_{g \in G} p_{t,g_{b,u}} + ens_{t,b} + \sum_{a_{b',b},y} f_{a,y,t} - \sum_{a_{b,b'},y} f_{a,y,t} = \\ &+ \sum_{z:FRT_{b,z}>0} \Big(\frac{frt_{t,b,z}}{FRT_{b,z}} \cdot FL_{t,b,z} \Big) + TL_{t,b} \quad \forall\, t \in T, \forall\, b \in B \end{aligned} \tag{6}$$

7.2 Generation Limits (Traditional)

The minimum and maximum generator output limits of traditional generator g at each time step t are given by

$$P^{min}_{g_{b,u}} \leq p_{t,g_{b,u}} \leq P^{max}_{g_{b,u}} \quad \forall\, t \in T, \forall\, g \in G^T \tag{7}$$

7.3 Generation Limits (Variable)

The minimum and maximum generator output limits of variable generator g at each time step t are given by

$$P^{min}_{g_{b,u}} \leq p_{t,g_{b,u}} \leq C_{t,g_{b,u}} \cdot P^{max}_{g_{b,u}} \quad \forall\, t \in T, \forall\, g \in G^R \tag{8}$$

Here $C_{t,g_{b,u}}$ represents the normalized output of g at time period t.

7.4 Transmission and Subtransmission Model

Transmission lines are divided into non-damaged lines which are in service initially and damaged lines. The first transmission equation sets a reference bus $b = B$ where the voltage angle is set to 0 for all timesteps t.

$$\theta_{t,B} = 0 \quad \forall\, t \in T \tag{9}$$

7.4.1 Non-Damaged Lines

Flow $f_{a_{b,b'},y,t}$ across arc a, branch id y, timepoint t is restricted by

$$F^{min}_{a,y} \leq f_{t,a,y} \leq F^{max}_{a,y} \quad \forall\, t \in T, \forall\, a \in A^{ND}, \forall\, y \in Y. \tag{10}$$

Ohm's Law for non-dammaged lines is represented by

$$X_{a_{b,b'},y} \cdot f_{t,a_{b,b'},y} = \theta_{t,b} - \theta_{t,b'} \quad \forall\ t \in T, \forall\ a \in A^{ND}, \forall\ y \in Y. \tag{11}$$

where $X_{a_{b,b'},y}$ represents the reactance of arc a, branch id y.

7.4.2 Damaged Lines - Recovery Candidates

Transmission line limits for damaged lines are given by

$$s_{t,a,y} \cdot F^{min}_{a,y} \leq f_{t,a,y} \leq s_{t,a,y} \cdot F^{max}_{a,y} \quad \forall\ t \in T, \forall\ a \in A^{D}, \forall\ y \in Y. \tag{12}$$

where $s_{t,a,y}$ is the status of the transmission line a, branch id y, in timestep t. Line status $s_{t,a,y}$ is defined as a binary variable $\in \{0, 1\}$, and is monotonically increasing.

$$s_{t,a,y} \geq s_{t',a,y}; \quad \forall t > t' \text{ for line } a \text{, branch id } y \tag{13}$$

The purpose of (13) is to ensure that once a line is restored ($s_{t,a,y} = 1$) during the simulation, it cannot be taken out of service again. Ohm's Law for damaged lines uses a disjunctive constraint to model powerflow

$$\begin{aligned} &- M \cdot (1 - s_{t,a,y}) \leq X_{a_{b,b'},y} \cdot f_{t,a_{b,b'},y} - (\theta_{t,b} - \theta_{t,b'}) \\ &\leq M \cdot (1 - s_{t,a,y}) \quad \forall\ t \in T, \forall\ a \in A^{D}, \forall\ y \in Y. \end{aligned} \tag{14}$$

7.5 Transmission and Subtransmission Recovery Budget Equations

The following section describes two options for restricting the amount of equipment that can be recovered in each time period. Option 1 (equation (16)) for transmission and subtransmission restricts the amount of work crew days that can be expended on recovery in any timestep to be less than a recovery time budget for that timestep $RB^{wcd}_{TS,t}$.

$$\sum_{a \in A^D, y \in Y} s_{t,a,y} \cdot TRT^{wcd}_{a,y} \leq RB^{wcd}_{TS,t} \quad \forall\ t \in T \tag{15}$$

$$\sum_{a \in A^D, y \in Y} trt^{wcd}_{t,a,y} \leq RB^{wcd}_{TS,t} \quad \forall\ t \in T \tag{16}$$

$$0 \leq trt^{wcd}_{t,a,y} \leq TRT^{wcd}_{a,y} \quad \forall\ t, a, y \tag{17}$$

$$trt^{wcd}_{t,a,y} \geq trt^{wcd}_{t',a,y} \quad \forall\ t > t', a, y \tag{18}$$

$$\frac{trt^{wcd}_{t,a,y}}{TRT^{wcd}_{a,y}} \geq s_{t,a,y} \quad \forall\ t, a, y \tag{19}$$

Option 2 (equation (20)) for transmission and subtransmission restricts the number of lines that can be recovered in any timestep to be less than a recovery time budget for that timestep $RB^{num}_{TS,t}$.

$$\sum_{a \in A^D, y \in Y} s_{t,a,y} \leq RB^{num}_{TS,t} \quad \forall\ t \in T \tag{20}$$

7.6 Substation (Bus) Outage Model

Bus status $\rho_{b,t}$ is treated as a continuous variable from 0-1 for buses that are initially deemed out of service (a station out of service is just multiple buses out of service).The approximation of bus status being continuous is done to reduce computational complexity. In reality, we would likely only consider a bus in service or out of service, not partially in service.

$$0 \leq \rho_{b,t} \leq 1 \quad \forall b, t \tag{21}$$

Additionally, it is assumed that the bus status $\rho_{b,t}$ must be increasing, or that once a portion of the recovery has been completed it cannot be degraded.

$$\rho_{b,t} \geq \rho_{b,t'} \quad \forall b, t > t' \tag{22}$$

An assumption for undamaged transmission lines connected to damaged buses is that they are disconnected at the begging of the recovery and assigned a recovery time of zero work crew days. For transmission lines that are offline because they are damaged no change is necessary. Additionally, at damaged buses, generators are forced offline until the bus repairs have begun and de-rated until the the bus has been fully recovered.

$$0 \leq p_{t,g_{b,u}} \leq \rho_{b,t} \cdot P^{max}_{g_{b,u}} \quad \forall b, t, g \tag{23}$$

Next we allow the station to recover fractionally, based on how many of its total connecting lines are recovered. For example, if there are 5 lines connected to a bus ($LC_b = 5$) and 2 of them recover for a particular time step t then the right hand side sums to 2 and $\rho_{b,t}$ must be 0.4 (the bus is 40 percent recovered).

$$LC_b \cdot \rho_{b,t} = \sum_{a_{b',b} \in A^D : b \in B^D, y} s_{t,a,y} + \sum_{a_{b,b'} \in A^D : b \in B^D, y} s_{t,a,y} \quad \forall b, t \tag{24}$$

Finally, a budget is placed on the amount of buses that can be recovered per time step.

$$\sum_b \rho_{b,t} \leq RB^{wcd}_{B,t} \quad \forall t \tag{25}$$

7.7 Feeder Model

For each feeder z at bus b, $FRT_{b,z}$ is used to estimate how much time it will take to recover the feeder in terms of work crew days. During the recover simulations, the decision variable $frt_{t,b,z}$ tracks the amount of labor that has been expended towards recovering the feeder. When $frt_{t,b,z}$ = 0, no recovery has occurred. However, when $frt_{t,b,z} = FRT_{b,z}$ the recovery of that feeder is complete. This is modeled by

$$0 \leq frt_{t,b,z} \leq FRT_{b,z} \quad \forall t, b, z. \tag{26}$$

Similar to transmission and sub-transmission, a recovery budget for feeders is implemented for each timestep t of the recovery. (27) restricts the total number of work crew days that can be recovered in any timestep to be less than a threshold $RB^{num}_{F,t}$ defined by an RSA analyst.

$$\sum_{b,z : FRT_{b,z} > 0} frt_{t,b,z} \leq RB^{num}_{F,t} \quad \forall\ t \in T \tag{27}$$

7.8 Thermal Generation Considering Black Start and Non-Black Start Units.

In the following section, the mathematical formulation is given for requiring a non-blackstart (NBS or equivalently BS^c) unit to recieve cranking power before it can energize. For simplicity the formulation is written for a generator bus with a single non-black start unit, at most one load, and a single line connecting the generator bus to the rest of the system. However, a more generic representation of components as the generator bus can be achieved with slightly more complex equations. When using the non-black start modeling feature, the equations in this section replace the equations in section 7.2.

Black start (BS) units are allowed to simply switch on according to

$$P^{min}_{t,g_{b,u,k}} \leq p_{t,g_{b,u,k}} \leq P^{max}_{t,g_{b,u,k}} \quad \forall\, t \in T, \forall\, g \in G^T \cap BS. \tag{28}$$

Non-black units, however, require cranking power before they can energize. Equation (29) models the two modes of operation of BS^c units. When $\eta_{t,g_{b,u,k}} = 0$, this forces BS^c units to operate as a load with an output of $-0.1 \cdot P^{max}_{t,g_{b,u,k}}$. However, if a unit's status changes to $\eta_{t,g_{b,u,k}} = 1$ this allows the unit to operate as generator with power output between $P^{min}_{t,g_{b,u,k}}$ and $P^{max}_{t,g_{b,u,k}}$.

$$\begin{aligned} & -0.1 \cdot P^{max}_{t,g_{b,u,k}} \cdot (1 - \eta_{t,g_{b,u,k}}) + \eta_{t,g_{b,u,k}} \cdot P^{min}_{t,g_{b,u,k}} \leq P_{t,g_{b,u,k}} \\ & \leq -0.1 \cdot P^{max}_{t,g_{b,u,k}} \cdot (1 - \eta_{t,g_{b,u,k}}) + \eta_{t,g_{b,u,k}} \cdot P^{max}_{t,g_{b,u,k}} \\ & \forall t, g \in G^T \cap BS^c \end{aligned} \tag{29}$$

Once a BS^c unit is turned on, it stays on which is captured by equation (30).

$$\eta_{t,g_{b,u,k}} \geq \eta_{t',g_{b,u,k}};\ \forall t > t' \text{ for } g_{b,u,k} \tag{30}$$

To measure if cranking power is received at buses with BS^c units, the power injection into the bus is measured with (31). For a given $g \in G^T \cap BS^c$ at bus b and time t, $a_{b,t} \geq 1$ indicates that cranking power is being received. For $a_{b,t} < 1$, power may be servicing a load at the bus but not enough to supply cranking power to the BS^c unit.

$$a_{b,t} = \frac{\sum_{a(b',b),y} f_{a'_{b,b},y,t} - \sum_{a(b,b'),y} f_{a'_{b,b},y,t}}{0.1 \cdot P^{max}_{t,g_{b,u,k}}} \forall\, t > 0, \forall\, b \text{ such that } \exists\, g_{b,u,k} \in G^T \cap BS^c \tag{31}$$

Because the unit only needs to receive cranking power in a single time step for it to turn on, logic is needed that (1) provides an indication of when cranking power has been received (Equation (32)) and (2) allows the unit to turn on ($\eta_{t',g_{b,u,k}} = 1$) once cranking power has been received (Equation (33)). A time shift between when a unit receives cranking power and when it can start producing power is necessary as (29) forces a unit to be either a load or generator in any given time step. It cannot be both. This does not need to be explicitly modeled though as $\eta^a_{b,t}$ will not change to 1 the first time period $max^a_{b,t} \geq 1$ to avoid infeasibility.

Thus, $max^a_{b,t}$ in (32) can be interpreted as an indicator variable that at each t indicates whether cranking power is being supplied or has been supplied at some point in the past to $g_{b,u,k} \in G^T \cap BS^c$.

$$max^a_{b,t} = max(0, a_{b,0}, a_{b,1}...., a_{b,t}) \quad \forall\, t,\ \forall\, b \text{ such that } \exists\, g_{b,u,k} \in G^T \cap BS^c \tag{32}$$

$$\eta^a_{b,t} \leq max^a_{b,t} \quad \forall\, t, \forall\, b \text{ such that } \exists\, g_{b,u,k} \in G^T \cap BS^c \tag{33}$$

While the max function in (32) is not a linear equation, it can be reformulated using mixed integer linear equations. The reformulation is given by (34)-(37). For this specific problem, the re-formulation is complicated by the need to compute the $max^a_{b,t}$ at every bus b containing a non-black start unit and time step t. To clarify how the reformulation of the max function is done, it is demonstrated for a single max with less indices in section 8.0.

$$max^a_{b,t} \geq a_{b,\tau} \quad \forall\ \tau \leq t, \forall\ b \text{ such that } \exists\ g_{b,u,k} \in G^T \cap BS^c \tag{34}$$

$$max^a_{b,t} \leq 0 + (1 - k^0_{b,t}) \cdot M' \quad \forall\ t, \forall\ b \text{ such that } \exists\ g_{b,u,k} \in G^T \cap BS^c \tag{35}$$

$$max^a_{b,t} \leq a_{b,\tau} + (1 - k_{b,t,\tau}) \cdot M' \quad \forall\ \tau \leq t, \forall\ b \text{ such that } \exists\ g_{b,u,k} \in G^T \cap BS^c \tag{36}$$

$$k^0_{b,t} + \sum_{\tau} k_{b,t,\tau} = 1 \quad \forall\ \tau \leq t, \forall\ b \text{ such that } \exists\ g_{b,u,k} \in G^T \cap BS^c \tag{37}$$

8.0 Appendix: Simplified Reformulation of a Max Function

(38) defines decision variable max^a as the maximum of decision variables $a_1, a_2, ..., a_N$. However this is a nonlinear constraint.

$$max^a = max(a_1, a_2, ..., a_N) \tag{38}$$

(39) - (41) reformulate (38) into a set of mixed integer linear constraints where k_n for $n = 1, 2, ...N$ are binary decision variables and M is a large number. When $k_n = 1$, this n represents the maximum a_n and (39) - (40) ensure max^a is equal to that a_n.

$$max^a \geq a_n \quad \text{for } n = 1, 2, ...N \tag{39}$$

$$max^a \leq a_n + (1 - k_n) \cdot M \quad \text{for } n = 1, 2, ...N \tag{40}$$

$$\sum_n k_n = 1 \tag{41}$$